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Homoclinic and heteroclinic solutions of upheaval buckling

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Upheaval of a heavy flexible strut from a rigid bed is viewed as an initial-value problem and spatial kinetic and potential energy functions are consequently defined. Upheaval from a perfectly flat state is characterized by the simultaneous vanishing of both functions at the boundaries. Smooth (flat to flat) connection without contact over a step in the bed is thus deemed impossible. A linear non-homogeneous fourth-order ordinary differential equation governs in regions of separation, but not when contact with the bed is maintained. This piecewise property is enough to ensure that several kinds of homoclinic and heteroclinic solutions exist for prop and step imperfections. Application is to subsea pipelines and examples of competing solutions for a realistically proportioned finite-length experimental pipe are included.

1. Introduction

The problem of uplift or upheaval of a heavy compressed strut from a rigid bed, as well as being of practical importance to the off-shore engineering industry (Hobbs 1984; Taylor & Gan 1987; Ju & Kyriakides 1988), has some intriguing mathematical properties. It responds well to analytical shooting methods (Blackmore 1995) and in the spirit of this special issue exhibits both single and multiple-hump localized solutions (Hunt & Blackmore 1996). In the sense sometimes referred to as the dynamical systems analogy (Kirchhoff 1859; Hunt *et al.* 1989), this structural boundary-value problem (BVP) is usefully interpreted as a dynamical system, with initial-value (IVP) information and spatial energy characteristics.

The conventional analytical approach is to model the system as a BVP of unknown length. For a perfectly flat bed of finite stiffness, in the absence of boundary effects, there would exist inside the bed a flat (fundamental) equilibrium state, where the force in the bed just compensates for the weight of the pipe. Linear eigenvalue analysis shows that deflection (y) away from this state is by oscillation with exponentially growing amplitude. As the bed stiffness approaches infinity, both the amplitude and the wavelength of the oscillation tend to zero, so upheaval from a rigid bed is characterized by a non-zero deflection growing from a flat state at each end. Although this seems to imply that y and all its derivatives are zero at the ends, deflection is brought into the pipe and lift-off achieved by reactive shear loads, giving non-zero third derivatives ($\dddot{y}$), at the boundaries; y and the remaining derivatives ($\dot{y}$ and $\ddot{y}$) remain zero however. Techniques such as the finite element method experience considerable difficulties associated with unknown buckle lengths.

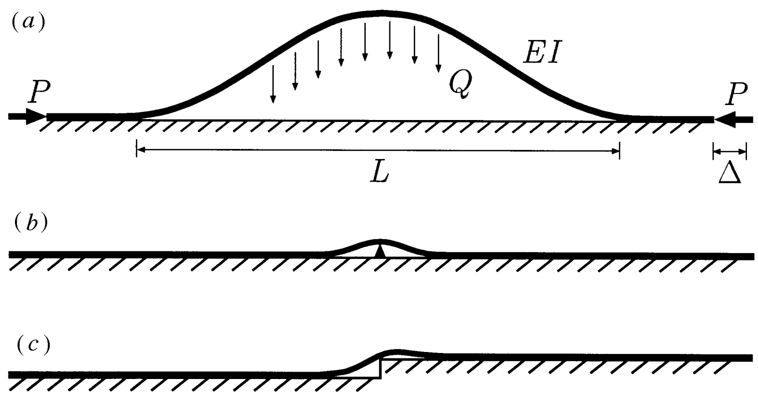

Figure 1. (*a*) Heavy pipe lifting from a flat bed. (*b*) Prop imperfection. (*c*) Step imperfection.

As a BVP the system is thus at first glance overdetermined. The uplifted shape can be characterized by a standard fourth-order linear ordinary differential equation (ODE), but is subject to three boundary conditions, $y = \dot{y} = \ddot{y} = 0$, at each end. This is traditionally resolved (Hobbs 1981; Taylor & Gan 1984) by including buckle length as an unknown and replacing three conditions at one end by two symmetric-section conditions, $\dot{y} = \dddot{y} = 0$, at the centre. However, the approach appears rather inflexible; it has difficulty in dealing with deflection over a step for example, where use of a symmetric-section condition is denied (Blackmore 1995).

For this and associated reasons we adopt shooting methods, whereby numerical integration of the governing ODE is undertaken with a set of four initial values (Champneys & Spence 1993). One or more of these can be varied until a target condition (e.g. the symmetric section as defined in § 3 *c* (iii)) is met at some previously unspecified point along the length; buckle lengths thus decide themselves. In this context, spatial analogies of kinetic and potential energy from dynamics prove most useful. Although the problem is driven by practical considerations, we take the opportunity to expand on some implications of modelling with a rigid bed. Solutions of regular periodic contact, or skipping, are identified, with homoclinic and heteroclinic connections emerging, depending on the type of imperfections, between both flat and periodic states. Alternative equilibrium states are interpreted for an experimental rig at present under development at Sheffield Hallam University (Tran 1994; Taylor & Tran 1996), aimed at exploring the relative effects of prop and step imperfections on subsea pipelines under compression due to hot oil or gas.

2. Practical considerations

The prototype system under consideration is shown in figure 1. Figure 1*a* has a finite-length inextensional elastic pipe of weight per unit length Q and bending stiffness EI, carrying a compressive load P, in a state of upheaval from a perfectly flat rigid bed. The separated length is denoted by L and the end-shortening (corresponding deflection of the load) by Δ. Upwards point load reactions (not shown) of magnitude $\frac{1}{2}QL$ occur at each lift-off point. Because Q acts to hold the pipe to the bed, an infinite (critical) load is required to buckle the perfect system of figure 1*a*. To describe the initiation of buckling more realistically, imperfections in the bed are therefore often introduced (Taylor & Gan 1986; Taylor & Tran 1993). Figures 1*b*,*c* show two typical types, the prop and the step. A shooting method applied to equa-

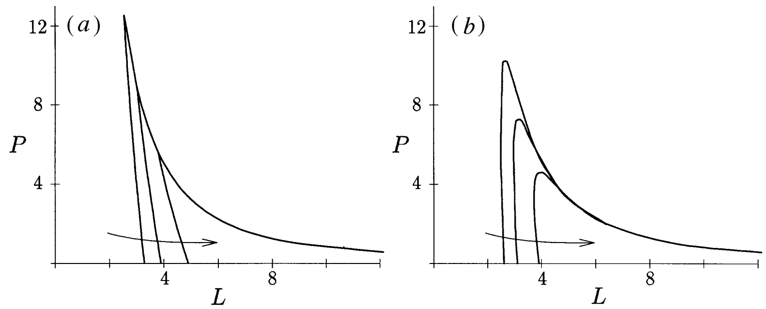

Figure 2. Comparative effects of (*a*) prop and (*b*) step imperfections. Arrows indicate increasing imperfection heights $h = 0.1$, 0.2 and 0.5.

tion (3.1) of the next section, with EI and Q each set to unity, leads to the response diagrams of figure 2, in which plots of P against L are shown for three different values for each imperfection.

Although it may not be strictly meaningful to compare quantitatively these two different types of imperfection, qualitative differences are plain to see. Each response goes through a maximum limiting load, with unbuckled states to the left (low L) and uplifted states to the right. However, the prop imperfection of figure 2*a* presents this as a sharp corner on each response curve, reflecting the fact that after upheaval contact with the prop is lost, whereas the step imperfection of figure 2*b* rounds the corners to genuine limit points as shown. It thus appears that a more profound destabilizing role can be attributed to the step than the prop.

Inherent in the concept of homoclinic connection is that of an infinite domain, with x extending to infinity in at least one direction as y approaches zero. Practical pipelines, on the other hand, are governed by 'active length', which friction dictates as being that which is capable of unloading into the buckled region; outside this length the pipe is essentially unaware that buckling has occurred. Active length is of considerable practical significance—in the presence of axial compressibility it governs how much effective shortening the uplifted region receives—but it changes as buckling progresses and is difficult to measure or estimate. Even conservative estimates put active lengths as greater than those of reasonably designed experiments, however, and we thus conclude that experiments of upheaval are unlikely to be free from boundary effects; these include skipping, in which the pipe periodically loses contact with the bed, as seen later. We note that shooting methods can be adapted for the practical case including friction, the associated forces being readily calculated as displacement progresses along the length (Blackmore & Hunt 1996).

3. Formulation

(*a*) *Differential equation*

The linearized governing equation for a strut of weight/unit length Q, bending stiffness EI and carrying an axial compression P (Timoshenko & Gere 1963) is,

$$EI\ddot{\ddot{y}} + P\ddot{y} + Q = 0, \tag{3.1}$$

where dots denote differentiation with respect to the spatial axial coordinate x. To generalize this equation and find its eigenvalues we write $p = P/\sqrt{EIQ/\ell}$, $\tilde{y} = y/\ell$

and redefine dots to denote differentiation with respect to $\tilde{x} = x\sqrt[4]{Q/EI\ell}$, to give

$$\ddot{\ddot{\tilde{y}}} + p\ddot{\tilde{y}} + 1 = 0, \tag{3.2}$$

where ℓ is a typical length measure, introduced merely to complete the non-dimensionalization; for all practical purposes, ℓ can be taken as unity. After writing $\tilde{y} = A\mathrm{e}^{\lambda\tilde{x}}$, the characteristic equation becomes

$$\lambda^4 + p\lambda^2 = 0, \tag{3.3}$$

which is of a degenerate form with two imaginary and two coincident zero eigenvalues. We note the similarity with the rotationally symmetric mode of the axially compressed cylinder (Lord *et al.*, this volume). The general solution is

$$\tilde{y} = a + b\tilde{x} + c\cos\sqrt{p}\tilde{x} + d\sin\sqrt{p}\tilde{x} - \frac{1}{2p}\tilde{x}^2, \tag{3.4}$$

where a, b, c and d are real constants which depend on boundary conditions (Hunt & Blackmore 1996). From this point, unless identified otherwise, all representations are dimensionless and we drop the tilde.

(*b*) *Spatial energies*

If equation (3.2) is viewed as a Lagrange equation, analogues to kinetic and potential energy can be written, respectively, as

$$\begin{aligned} T &= \dot{y}\dddot{y} + \tfrac{1}{2}p\dot{y}^2, \\ V &= y - \tfrac{1}{2}\ddot{y}^2. \end{aligned} \tag{3.5}$$

Whilst the Lagrangian $\mathcal{L} = T - V$ for such a system is not unique, this choice proves particularly convenient; these spatial energy functions are used extensively in the analysis that follows. The Hamiltonian nature of the system implies that $\mathcal{H} = T + V = \text{const.}$ over the spatial domain described by x.

(*c*) *Initial/boundary conditions*

No immediate distinction is made in the general description below of initial and boundary conditions applied to the above differential equation. However, a fourth-order system, when treated as a BVP, naturally suits two prescribed boundary conditions at each end. We identify in what follows a number of archetypal end conditions of practical importance which, depending on the type of solution under consideration, appear in different combinations as initial and target conditions for the numerical shooting method. Among these is the asymptotic boundary, which necessarily involves three separate conditions. Such a boundary is clearly more suited to IVP than BVP analysis, and this reinforces the selection of a shooting method based on a Runge–Kutta formulation, rather than a standard structural approach such as the finite element method.

(i) *Asymptotic condition*

If upheaval is from a flat rigid bed, the point of separation, which we choose to designate $y = 0$, is marked by the three conditions $y = \dot{y} = \ddot{y} = 0$. If, in addition, the vertical reactive force at lift-off is obliged to be upwards, this is accompanied by the further condition, $\dddot{y} > 0$.

By analogy with the system on the finite stiffness bed (Hunt & Blackmore 1996),

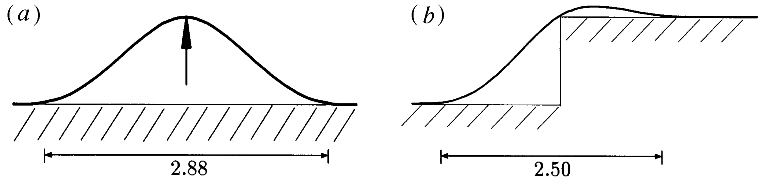

Figure 3. Draped solutions to non-dimensional equation (3.2) with $\ell = 1$, plotted at $p = 6$ over (a) prop of height $h = 0.1$; (b) step of height $h = 0.1$ (upheaval lengths are as indicated).

although it is met in finite rather than infinite x, we continue to refer to this as an *asymptotic condition*. It applies at the ends of a homoclinic connection. With y measured from the bed, the spatial energy functions (3.5) give $T = V = \mathcal{H} = 0$ at an asymptotic boundary point. Similar conditions also apply at the boundaries of a heteroclinic connection between different energy levels over a step, with $y \neq 0$ (and hence $\mathcal{H} \neq 0$) at one of the levels.

(ii) *Fixed-anchor point (FAP) condition*

For a fixed-anchor point or clamped condition, $y = \dot{y} = 0$. Also, for upwards lift-off from a flat bed, $\ddot{y} > 0$ at this point. When the bed is flat and level these conditions also apply at points of contact in the skipping mode (see § 3 *d* (iv)). Unlike point support at a corner (see figure 4 for instance), $\dot{y}$ is necessarily zero and a jump in $\dddot{y}$ due to the reactive force cannot then affect the total spatial energy $\mathcal{H}$, which is thus conserved through a skipping point of contact. Under experimental and certain practical conditions, FAPs exist where a pipe is clamped or restrained. At such a point, $T = 0$ and $\mathcal{H} = V = \frac{1}{2}\ddot{y}^2$.

(iii) *Symmetric-section condition*

Points on the so-called *symmetric-section* (Hunt & Wadee 1991) arise at the centre of an uplifted region over a flat bed. At such a point where $y = y_{\max}$, $\dot{y} = \dddot{y} = 0$ and hence $T = 0$ and $\mathcal{H} = V = y_{\max} - \frac{1}{2}\ddot{y}^2$.

(*d*) *Primary solutions*

More complicated shapes or behaviours can arise from combinations or modifications of the following forms, which again seem to carry some primary significance.

(i) *Draped shapes*

At zero or low axial loads, a pipe will drape over a prop or step as shown in figures 3*a*,*b*, respectively. In figure 3*a*, asymptotic conditions can only be met at both ends if the pipe rests symmetrically about the prop, such that the jump in $\dddot{y}$ at the prop does not result in a jump in the total spatial energy $\mathcal{H}$. To satisfy asymptotic conditions at each level of figure 3*b*, however, the pipe must lean against the step, as described later in § 3 *d* (iii).

(ii) *Homoclinic connection*

For a pipe resting initially on a prop, as load is increased equilibrium states can be found in which the pipe has lost contact with the prop. The deflected shape after upheaval then connects an asymptotic point to itself; with no possibility of input of energy, the hamiltonian $\mathcal{H}$ is conserved over the full range of x. While noting again that, in contrast to the classical form for a bed of finite stiffness, this occurs over a finite distance, we shall continue to refer to it as a *homoclinic connection*.

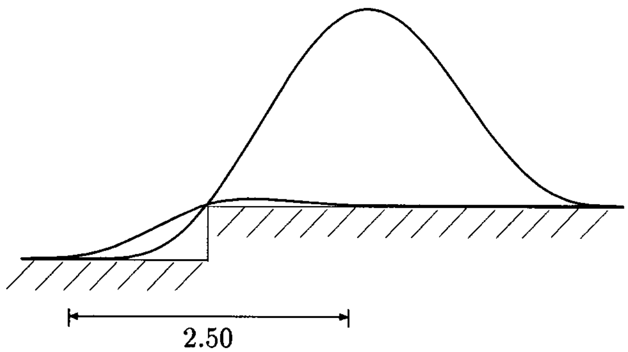

Figure 4. Pre- and post-upheaval leaning solutions to non-dimensional equation (3.2), with $\ell = 1$ at $p = 6$ and step height $h = 0.1$.

The concept of active length means that stable irregularly spaced multi-humped solutions can exist, with each hump outside the range of influence of the others. In the absence of friction stability is lost (see Sandstede, this volume), so if two humps are within range of one another, stability becomes an open question.

(iii) *Leaning solutions*

Analogously to the draped and homoclinic solutions over the prop, as the load is raised two equilibrium possibilities appear supported by a step, as shown in figure 4. In each case, the non-zero reaction at the corner of the step implies a jump in $\dddot{y}$, which couples with the non-zero $\dot{y}$ to give a jump in T. Enough spatial energy $\mathcal{H}$ is thus added or removed to enable asymptotic conditions to be satisfied at each level. These are *heteroclinic connections*, between different flat equilibrium states.

(iv) *Skipping solutions*

For a perfectly flat bed, skipping solutions exist that pass though FAP boundary conditions each time the pipe touches the bed. From the general solution (3.4), setting $y = \dot{y} = 0$ at $x = 0$ gives, directly,

$$a = -c = \frac{p\ddot{y}_0 + 1}{p^2} \quad \text{and} \quad b = -d\sqrt{p} = \frac{\dddot{y}_0}{p}. \tag{3.6}$$

However, $\ddot{y}_0$ and $\dddot{y}_0$ are not independent, but are linked by the symmetry of the flat bed. Symmetric-section conditions $\dot{y} = \dddot{y} = 0$ applied at $x = \frac{1}{2}L$ (half the skiplength) give

$$\tan\sqrt{p}\tfrac{1}{2}L = d/c \quad \text{and} \quad b = L/2p. \tag{3.7}$$

From (3.6), skiplength L is then

$$L = 2\dddot{y}_0, \tag{3.8}$$

while second derivatives $\ddot{y}_0$ and $\ddot{y}_{L/2}$ can be written in terms of $\dddot{y}_0$ as

$$p\ddot{y}_0 = \frac{\sqrt{p}\dddot{y}_0}{\tan\sqrt{p}\dddot{y}_0} - 1, \tag{3.9}$$

$$p\ddot{y}_{L/2} = \frac{\sqrt{p}\dddot{y}_0}{\sin\sqrt{p}\dddot{y}_0} - 1. \tag{3.10}$$

Small $\dddot{y}_0$ leads to negative $\ddot{y}_0$ and contravenes the FAP conditions, so vanishingly small amplitude skipping solutions are disallowed. Since $T_0 = T_{L/2} = 0$, $V_{L/2} = V_0$

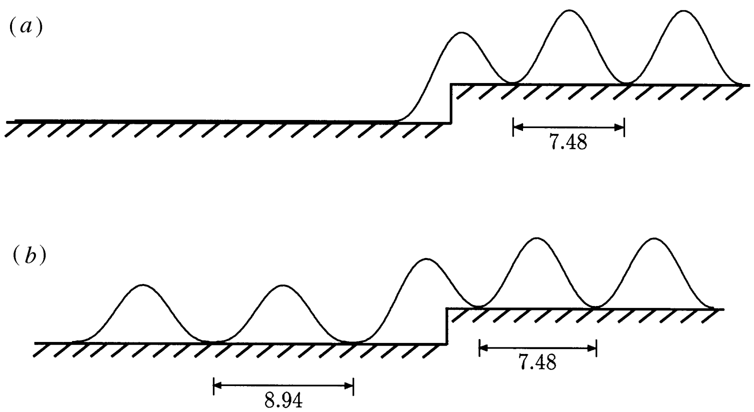

Figure 5. Heteroclinic connections involving skipping: (*a*) from a flat equilibrium state; (*b*) from one skipping state to another. Plotted for non-dimensional equation (3.2), with $\ell = 1$ at $p = 1$ and step height $h = 10$. Note that waves that differ only slightly on the upper level here connect with significantly different responses on the lower level.

and skipheight $y_{L/2}$ follows directly from the spatial potential energy V (3.5),

$$y_{L/2} = \tfrac{1}{2}(\ddot{y}_{L/2}^2 - \ddot{y}_0^2). \tag{3.11}$$

A skipping solution is typified by $\ddot{y}$ remaining continuous and positive at a point of contact, accompanied by a positive jump in $\dddot{y}$ associated with an upwards vertical reaction. For a perfectly flat bed, each skip is symmetrical about its centre and third derivatives $\dddot{y}$ at take-off and landing are equal and opposite; as only upwards reactions are allowed the first is necessarily positive and the second negative. Reactive forces at all points of contact are thus the same and complete regularity of behaviour (periodicity) is achieved.

Although the hamiltonian $\mathcal{H}$ is conserved over skipping points of contact, the very possibility of their presence provides other ways, apart from leaning, for a system to accommodate the jump in $\mathcal{H}$ associated with passage over a step. This is shown in figure 5, which demonstrates two valid forms of heteroclinic connection, from the flat fundamental state to a periodic skipping state, and from one skipping solution to another of a different frequency.

4. Characteristics of the step imperfection

Much work, experimental and theoretical, has been done on symmetric (prop) imperfections of various kinds (see, for example, Boer *et al.* 1986; Ju & Kyriakides 1988; Taylor & Tran 1993). Asymmetric imperfections like the step have received significantly less attention, however, despite the qualitatively more significant destabilizing role brought out in figure 2. Greater variety of solution is also possible. To demonstrate this we take the dimensions of a typical finite-length experimental rig with a step imperfection and impose upon it several of the possible solutions alluded to above.

(*a*) *Competing solutions*

For systems of finite length, or those governed by short active lengths, at least three types of solution compete for the most effective means of relieving a compressed state over a step. The leaning solution of figure 4 is one way of overcoming the difference in total spatial energy $\mathcal{H}$ between the levels. Asymptotic conditions can be satisfied at each point of contact and a heteroclinic connection established between the two flat equilibrium states. This is a truly localized solution, with the spatial position in x being governed entirely by the position of the step.

On the other hand, by skipping a pipe can carry a non-zero bending moment and hence a non-zero value of $\mathcal{H}$ back to a fixed anchor point. It is thus possible to connect over a step between an asymptotic boundary at a lower level and a skipping solution at an upper level, as seen in figure 5*a*, but not vice versa. (With y measured from the upper level, $\mathcal{H} = T + V = 0$, so at the lower level where y is negative and $T = 0$, $\ddot{y}^2 < 0$ from (3.5).) We refer to this as a semilocalized solution, heteroclinic between a flat and a periodic equilibrium state. Its spatial position in x depends on the position of the step only in the conditional sense that contact must be avoided, but is fixed by the distance from the FAP boundary.

Or finally, it may suit the boundaries and wavelengths involved for skipping solutions to apply on both sides of a step. As a skipping solution meets a step its periodicity is interrupted; periodicity at different wavelengths exists on each side of a step, as seen in figure 5*b*. Reactions closest to the step are both different from one another and from those of each periodic sequence. If FAP conditions are used at both ends of a relatively short pipe (as below) the problem is readily formulated in boundary-value terms. If, however, skipping is involved on each side of the step, different solutions with differing skiplengths may present themselves. Interesting mathematical questions, left for future study, are raised by long-but-finite lengths.

(*b*) *Experimental rig*

The following results are interpreted specifically for an experimental apparatus under development at Sheffield Hallam University under the direction of Dr N. W. Taylor. A steel pipe of wall thickness 1.6 mm and diameter 9.53 mm, 6 m in length, carries water at a slow flow rate. The ends are clamped so that all movement is restricted and FAP boundary conditions apply, and a 17 mm step is introduced at the mid-length. The water can be heated externally in a separate bath to induce buckling. The rig is an adaptation of one previously used for the study of prop imperfections (Taylor & Tran 1993; Tran 1994).

(*c*) *Friction and axial compressibility*

In genuine applications, friction between a pipe and the seabed, and axial compressibility, are clearly of prime significance. Via the concept of active length they together dictate how much elastic energy is available to unload into the uplifted region and hence determine the extent of the instability before restabilization.

For the effects of friction, consider the response curves of figure 6*a*. Here, for the dimensions of the above pipe and step height but over an active length governed by an assumed coefficient of friction $\phi = 0.2$, the variation of the load in the uplifted region, P, for the fully localized (leaning) solution is compared with that for the *control load*, P_0, found outside the active length. The latter varies linearly with temperature, while the difference in the two loads is taken up by friction. The curve for P has the form of figure 2*b* and continues to fall with L, but that for P_0 restabilizes, such that the indicated snap buckle at constant P_0 leads to a stable uplifted shape of just under 6 m in length—of the order of, but less than, the experimental length of 6 m; typically,

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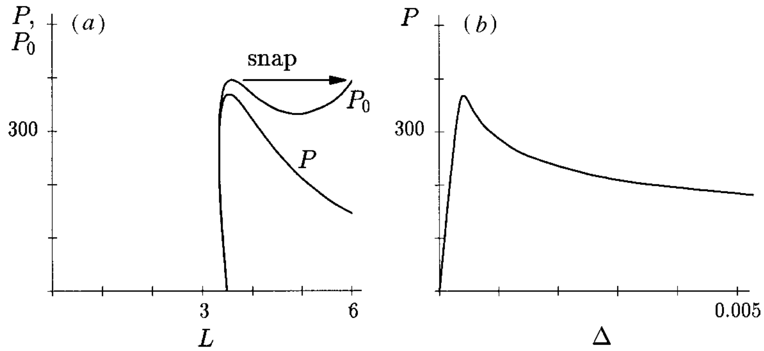

Figure 6. Response curves for the leaning solution. (*a*) Effects of friction under temperature control (active length *ca.* 100 m). (*b*) Load P plotted against end-shortening Δ under frictionless conditions (length = 6 m, constrained ends).

however, active lengths are found to be much greater than that of the experiment, of the order of 100 m at the destabilizing limit point for instance.

This suggests that practice and experiment may lead to quite different responses. The snap buckle of figure 6*a* depends critically on the axial compression over a long length unloading into the buckled region, and yet a typical experimental length is only of the order of the buckled length. Figure 6*b* shows load P plotted against the *total* end-shortening Δ for a frictionless bed and the length and step height in question. The latter comprises two components, EA-dependent shortening due to direct axial compression and EI-dependent shortening due to buckling. The axial (EA) component tends to tilt the curve to the right, with the degree of tilt increasing with total (active) length; the longer the pipe, the less stiff the initial response and the smaller the slope of the pre-buckling (fundamental) equilibrium path. The effect of this initial squash is to 'spring load' the system, such that *snap-back*, as exhibited, for example, by the axially compressed cylindrical shell (see figure 1 of Lord *et al.*, this volume), becomes a possibility. The horizontal snap buckle of figure 6*a* and consequent drop in P is then most closely represented, on the curve of figure 6*b*, by a downwards snap at constant Δ. Again, the longer the pipe, the greater the tilt on the curve and the more severe the snap-back.

For the length and step height considered here, however, the phenomenon of snap-back is clearly absent; the experimental length is simply too short. We hence conclude that the experimental leaning response, in contrast to that expected from a pipeline of practical dimensions laid over a step, is likely to be stable and undramatic.

The effect of end-constraint, either by friction or by clamping, is thus to inhibit the extreme instability nominally met by the compressed strut of infinite length; the infinite compressive strain energy available for transmission into the buckled region is rendered finite. After the maximum on the response curve, a jump may or may not take place to a reduced load state, the extent or otherwise of the jump depending crucially on either the active, or the actual experimental, length.

(*d*) *Comparative results*

Figure 7*a* compares fully and semilocalized solutions for the experimental apparatus of interest. The response curves are plotted up to the maximum possible length of 6 m, but, of course, for these localizing solutions they apply also for shorter lengths. In the loading region only the leaning solution exists, but in the post-maximum unloading region both solutions apparently are valid. However, figure 7*b* demon-

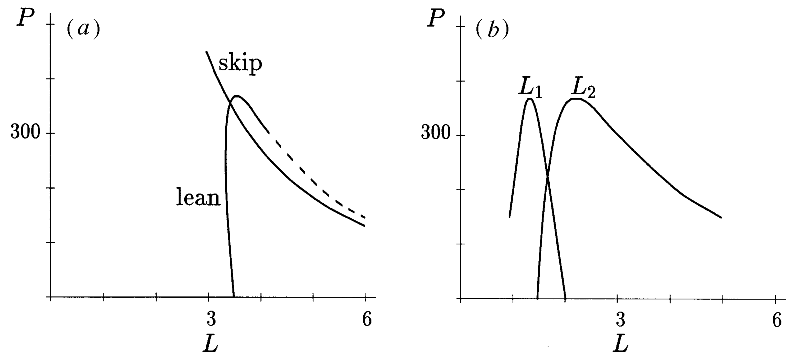

Figure 7. Response curves for the experimental data of § 4 *b*. (*a*) Comparison of fully localized (leaning) and semilocalized solutions. (*b*) Lower (L_1) and upper (L_2) separated lengths for the leaning solution.

strates a further constraint on the leaning solution, in that to maintain contact with the corner, the upper separated length L_2 must be less than the half length of 3 m; this invalidates the dashed portion of figure 7*a*. Although sought no further here, transitional solutions must exist in this region to transfer the path from the fully to the semilocalized state.

The localized solutions of figure 7 are valid for all lengths up to 6 m. Standard boundary-value solutions for fixed 6 m lengths are also available. Generally, skipping either on the upper level alone or on both the upper and lower levels provides further possibilities. For the length and load levels of interest here such responses complicate matters only at the higher loads of the 'skip' curve of figure 7*a*, where skiplengths are less than the upper length of 3 m. For moderately longer pipes, skipping becomes more of an option, with the number of possibilities for solution increasing with the experimental, or active, length. We note also that active lengths themselves could be considerably modified by skipping effects.

Actual detailed experiments are yet to be performed on the pipe in question and other factors would of course be involved in the choice of solution, not least imperfections in the pipe itself. However, on increasing temperature we hope to be able to identify more than one post-upheaval equilibrium state and, in particular, the transition from fully to semi- or unlocalized shape, as the limit of the (leaning) response curve of figure 7*a* is met. For longer experimental lengths, or the active lengths found in practice, full localization rather than skipping would be expected to provide the least stiff, and hence most likely, of the available options.

5. Concluding remarks

The paper takes a standard formulation for upheaval buckling and applies it in two new directions. The first is the study of the effects of asymmetric bed imperfections, typified by a step, rather than symmetric imperfections such as a prop of infiltrated material between the pipe and the bed (Taylor & Tran 1993). The extra significance of asymmetry is successfully demonstrated in figure 2. From a practical viewpoint it might be supposed that changes in seabed level are as fundamental to pipe modelling as the infiltration of foreign material, yet previous work relates almost exclusively to symmetric imperfections. The point is also made that as a source of bifurcation phenomena, asymmetry is again a richer pasture than symmetry.

The second, not entirely unconnected, new development is to explore the effects of finite lengths. Skipping solutions are computed for the first time and alternative solutions involving either asymptotic or fixed anchor point (FAP) conditions are compared for a typical experimental rig. A range of lengths and loads is identified over which a transition from fully localized to semilocalized behaviour must take place.

Finally, the dynamical systems analogy again proves most useful in identifying possible forms of localized, semilocalized and unlocalized periodic (skipping) solutions. We note that all the results presented here depend on the bed being flat and, apart from the step itself, level. For systems supported at finite positions rather than by a level bed—the so-called 'bed of nails' (Blackmore 1995)—many of the conclusions developed herein will not apply. More like the corner of a step than a flat bed, the latter allows non-zero first derivatives $\dot{y}$ and hence changes of spatial kinetic energy at discrete points of contact.

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